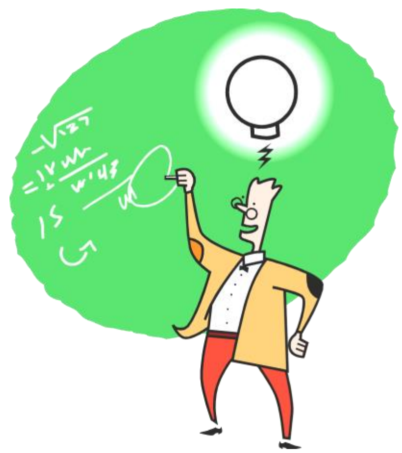
CH 28 – PREPARING FOR MORE QUADRATICS

❑ *INTRODUCTION*



In Chapter 13, we solved equations like $x^2 + 8x + 15 = 0$ by factoring:

$$x^2 + 8x + 15 = 0$$

$$\Rightarrow \quad (x + 5)(x + 3) = 0$$

$$\Rightarrow \quad x + 5 = 0 \text{ or } x + 3 = 0$$

$$\Rightarrow \quad \boldsymbol{x = -5 \text{ or } x = -3}$$

This special kind of equation, where the variable is squared (and may very well have two solutions), has a special name: We call it a ***quadratic equation***. Unfortunately, in the real world the probability of a quadratic expression being factorable is virtually nil. How do we remedy this situation? We develop two other methods to solve quadratic equations. This chapter will help you prepare for learning these methods in Chapters 29 and 30.

❑ *QUADRATIC EQUATIONS*

Let's look at the quadratic equation $w^2 = 7w + 18$. Because the variable w is squared, this is an example of a ***quadratic equation*** (from "quadrus," Latin for a 4-sided *square*). We can write this quadratic equation in **standard form** by subtracting $7w$ from each side of the equation and then subtracting 18 from each side of the equation:

$$w^2 - 7w - 18 = 0 \longleftarrow$$ **A quadratic equation in *standard form***

Notice that the quadratic (squared) term is written first, the linear term (the one with the variable to the 1st power) comes next, and the constant term comes last; the right side of the equation is 0. We also note that the coefficient of the squared term is 1, the coefficient of the linear term is –7, and the constant term is –18.

> The ***coefficient*** tells us how many of a particular thing we have. If we have $9n$, then we have 9 n's, and so the coefficient of $9n$ is 9. The coefficient of the term $-7xy$ is -7, and the coefficient of $13w^2$ is 13.
>
> Sometimes the coefficient is "implied." For example, the coefficient of u^2 is 1 (since $u^2 = 1u^2$), and the coefficient of $-m$ is -1 (since $-m = -1m$).

We can solve some quadratic equations in our head. For instance, consider the quadratic equation

$$x^2 = 25$$

What number squared equals 25? This is very important, so pay attention. There are two solutions to this equation: 5 and –5. After all, $5^2 = 25$, and $(-5)^2 = 25$. We conclude that the solutions of the quadratic equation $x^2 = 25$ are $x = 5$ and $x = -5$. So, as we study this chapter and the next two, keep in mind that a quadratic equation may (or may not) have two different solutions.

❑ *THE GENERAL QUADRATIC EQUATION*

Consider a general quadratic equation in **standard form**:

$$ax^2 + bx + c = 0$$

where x represents the variable (the unknown) and a, b, and c are numbers. Notice that the squared term is first, the linear term is second, the constant term is third, and there's a 0 on the right side of the equation. The examples below show some quadratic equations along with their corresponding values of a, b, and c:

$3x^2 - 7x + 10 = 0$ ⇨	$a = 3$	$b = -7$	$c = 10$
$y^2 - 9 = 0$ ⇨	$a = 1$	$b = 0$	$c = -9$
$-4z^2 + 19z = 0$ ⇨	$a = -4$	$b = 19$	$c = 0$
$2w^2 = 8 - 5w$ ⇨	$a = 2$	$b = 5$	$c = -8$

[To find the values of a, b, and c for this last equation, we need to rewrite the quadratic equation in standard form: $2w^2 + 5w - 8 = 0$, from which the values of a, b, and c can be determined.]

Homework

1. For each quadratic equation, first make sure that it's in standard form ($ax^2 + bx + c = 0$), then determine the values of a, b, and c:

a. $3x^2 + 9x + 17 = 0$

b. $2n^2 - 8n + 14 = 0$

c. $6y^2 + 2y - 2 = 0$

d. $5t^2 - 13t - 1 = 0$

e. $12a^2 + 13 = 0$

f. $x^2 - 13x = 0$

g. $u^2 - u + 1 = 0$

h. $-2w^2 - 19 = 0$

i. $-w^2 + 14w = 0$

j. $-z^2 - 99 = 0$

k. $2x^2 = 4x + 3$

l. $-3y^2 - 2y = -5$

m. $c^2 + 4 = 7c$

n. $6m^2 = 1 + m$

o. $18k^2 = 0$

p. $-3x^2 = -3x$

❑ ORDER OF OPERATIONS

Remember that exponents are done before multiplication, which itself comes before any addition or subtraction. For example,

	$12^2 - 4(2)(3)$	
$=$	$144 - 4(2)(3)$	(exponent first)
$=$	$144 - 24$	(multiplication second)
$=$	**120**	(subtraction last)

For a second example,

	$(-9)^2 - 4(3)(-5)$	
$=$	$81 - 4(3)(-5)$	(exponent first)
$=$	$81 - (-60)$	(multiplication second)
$=$	$81 + 60$	(subtracting a negative is adding)
$=$	**141**	(addition last)

❑ SQUARE ROOTS

$\sqrt{36} = 6$

$-\sqrt{144} = -12$

$\sqrt{1} = 1$

$\sqrt{0} = 0$

$\sqrt{225} = 15$

$\sqrt{50} \approx 7.0711$

$\sqrt{-9}$ does not exist (in this class)

is approximately equal to

❑ OPPOSITES

Recall that the *opposite* of N is $-N$. So we note the following:

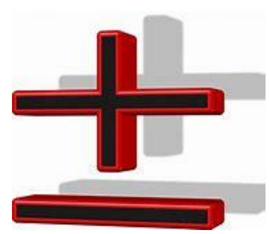
1) The opposite of a positive number is a negative number; for example, $-(+7) = -7$. Thus, if $b = 7$, then $-b = -7$.

2) The opposite of a negative number is a positive number; for example, $-(-12) = 12$. Thus, if $b = -12$, then $-b = 12$.

3) The opposite of a squared quantity (that isn't 0) is negative; for example, $-9^2 = -81$. On the other hand, don't forget that $(-9)^2 = 81$.

4) The opposite of 0 is 0.

❑ PLUS/MINUS



The symbol " ± " is read "***plus or minus***" and is just a concise way of indicating two numbers at once. For example, if you want to indicate the two numbers 7 and -7, you may write just ± 7. Another example might be $\pm\sqrt{81}$, which stands for the two numbers 9 and -9. But $\pm\sqrt{0}$ would only be 0, since $+0$ and -0 are really just two ways to express 0.

The equation $x^2 = 25$ from a few pages back yielded the fact that x could be 5 or -5, so we could write the solution as $x = \pm 5$.

Let's work out a specific problem containing the "plus or minus" sign.

Simplify: $\dfrac{10 \pm \sqrt{25}}{5}$

Taking the square root yields $\dfrac{10 \pm 5}{5}$.

Using the plus sign, we get $\frac{10+5}{5} = \frac{15}{5} = \mathbf{3}$,

while using the minus set yields $\frac{10-5}{5} = \frac{5}{5} = \mathbf{1}$.

Thus, the expression $\frac{10 \pm \sqrt{25}}{5}$ is merely a strange way to represent the numbers 3 and 1.

Homework

2. Evaluate each expression:

a. $13^2 - 4(3)(2)$ b. $0^2 - 4(2)(-1)$

c. $(-3)^2 - 4(-1)(-2)$ d. $(-5)^2 - 4(1)(0)$

e. $0^2 - 4(17)(0)$ f. $(-4)^2 - (-4)(-5)$

3. Evaluate each expression:

a. $\sqrt{49}$ b. $-\sqrt{6+3}$

c. $\sqrt{16}+\sqrt{9}$ d. $\sqrt{-25}$

e. $\sqrt{(-32)(-2)}$ f. $\sqrt{(-5)^2 - 4(1)(6)}$

g. $\sqrt{6^2 - 4(1)(9)}$ h. $\sqrt{(-10)^2 - 4(8)(2)}$

i. $\sqrt{1^2 - 4(1)(1)}$ j. $\sqrt{(-7)^2 - 4(4)(-2)}$

4. Evaluate each expression (the leading minus sign represents the *opposite* of the quantity which follows it):

a. $-(13)$ b. $-(-5)$ c. $-(-(-7))$

d. $-(+7)$ e. $-(-1)$ f. $-(0)$

5. What number or numbers are represented by each of the following?

a. ± 121 b. $\pm\sqrt{121}$ c. $\pm\sqrt{0}$ d. $\pm\sqrt{-1}$

6. Simplify each expression:

a. $100+\sqrt{49}$ b. $-5-\sqrt{121}$ c. $\frac{3+\sqrt{16}}{2}$ d. $\frac{-8-\sqrt{4}}{10}$

e. $7-\sqrt{49}$ f. $-10+\sqrt{100}$ g. $\frac{4\pm\sqrt{16}}{6}$ h. $\frac{-1\pm\sqrt{81}}{9}$

7. Evaluate the expression $\boldsymbol{b^2 - 4ac}$ for the given values of a, b, and c:

a. $a=-10,\ b=-5,\ c=3$ b. $a=4,\ b=6,\ c=-1$

c. $a=5,\ b=0,\ c=7$ d. $a=-2,\ b=-3,\ c=0$

8. Evaluate the expression $\pm\sqrt{\boldsymbol{b^2-4ac}}$ for the given values of a, b, and c:

a. $a=2,\ b=-7,\ c=-15$ b. $a=9,\ b=30,\ c=25$

c. $a=1,\ b=0,\ c=-81$ d. $a=2,\ b=-3,\ c=0$

❑ *THE ULTIMATE EXAMPLE*

Let's evaluate the expression

$$\frac{-b\pm\sqrt{b^2-4ac}}{2a} \qquad \text{for the values } a=1,\ b=-5,\ c=6$$

First change every variable to a pair of parentheses. This is not required, but it's a handy little trick to help us avoid errors:

$$= \quad \frac{-(\ \ )\pm\sqrt{(\ \ )^2-4(\ \ )(\ \ )}}{2(\ \ )}$$

$$= \frac{-(-5) \pm \sqrt{(-5)^2 - 4(1)(6)}}{2(1)}$$ (substitute the given values)

$$= \frac{5 \pm \sqrt{25 - 24}}{2}$$ (do exponents and multiplying)

$$= \frac{5 \pm \sqrt{1}}{2}$$ (finish the inside of the radical)

$$= \frac{5 \pm 1}{2}$$ (the positive square root of 1 is 1)

$$= \frac{5+1}{2} \text{ or } \frac{5-1}{2}$$ (split the plus/minus sign)

$$= \frac{6}{2} \text{ or } \frac{4}{2}$$ (simplify the numerators)

$$= \boxed{3 \text{ or } 2}$$ (and finish up)

There's no homework for this section, but don't be ☹ — there will be plenty of these problems when you study the Quadratic Formula two chapters from now.

❑ Review of Factoring Perfect Square Trinomials

Recall the term "perfect square." The number 100 is a perfect square because 100 can be written as the square of a number, namely $100 = 10^2$. One of the steps in solving a quadratic equation by completing the square is factoring a ***perfect square trinomial***. Let's look at four examples.

<u>Ex 1:</u>

$x^2 + 14x + 49$ is a *perfect square trinomial* because it factors into the square of a binomial:

$$x^2 + 14x + 49 = (x + 7)(x + 7) = \mathbf{(x + 7)^2}$$

Ex 2:

$n^2 - 20n + 100$ is a perfect square trinomial:

$$n^2 - 20n + 100 = (n - 10)(n - 10) = \mathbf{(n - 10)^2}$$

Now a couple of examples with fractions:

Ex 3: $a^2 + 3a + \frac{9}{4} = \left(a + \frac{3}{2}\right)\left(a + \frac{3}{2}\right) = \left(\boldsymbol{a + \frac{3}{2}}\right)^{\mathbf{2}}$

Check:

$$\left(a + \frac{3}{2}\right)^2 = \left(a + \frac{3}{2}\right)\left(a + \frac{3}{2}\right) = a^2 + \frac{3}{2}a + \frac{3}{2}a + \frac{9}{4} = a^2 + 3a + \frac{9}{4} \checkmark$$

Ex 4: $y^2 - \frac{2}{5}y + \frac{1}{25} = \left(y - \frac{1}{5}\right)\left(y - \frac{1}{5}\right) = \left(\boldsymbol{y - \frac{1}{5}}\right)^{\mathbf{2}}$

Check:

$$\left(y - \frac{1}{5}\right)\left(y - \frac{1}{5}\right) = y^2 - \frac{1}{5}y - \frac{1}{5}y + \frac{1}{25} = y^2 - \frac{2}{5}y + \frac{1}{25} \checkmark$$

Homework

9. Factor each perfect square trinomial:

a. $x^2 + 10x + 25$

b. $y^2 - 18y + 81$

c. $a^2 + a + \frac{1}{4}$

d. $m^2 - \frac{4}{3}m + \frac{4}{9}$

e. $z^2 + \frac{2}{5}z + \frac{1}{25}$

f. $w^2 - \frac{5}{3}w + \frac{25}{36}$

g. $b^2 + \frac{9}{5}b + \frac{81}{100}$

h. $u^2 + \frac{3}{2}u + \frac{9}{16}$

i. $n^2 - \frac{4}{7}n + \frac{4}{49}$

j. $x^2 + \frac{10}{11}x + \frac{25}{121}$

❑ THE SQUARE ROOT THEOREM

Let's look at the quadratic equation $x^2 = 100$. One way to solve it is by factoring:

$$x^2 = 100 \Rightarrow x^2 - 100 = 0 \Rightarrow (x + 10)(x - 10) = 0$$

$$\Rightarrow x + 10 = 0 \text{ or } x - 10 = 0 \Rightarrow x = -10 \text{ or } x = 10$$

The solutions of the quadratic equation $x^2 = 100$ are simply $\boldsymbol{x = \pm 10}$.

But there's an easier way to solve a quadratic equation like $x^2 = 100$. Just take the square root of each side of the equation, **remembering that the number 100 has <u>two</u> square roots**, namely 10 and –10. Therefore, $\boldsymbol{x = \pm 10}$, and no factoring is required.

For another example, let's solve the quadratic equation $n^2 = 30$. Remembering that **30 has two square roots**, we calculate the solutions of the quadratic equation to be $\boldsymbol{n = \pm\sqrt{30}}$.

For a third example, where we will need to simplify the radical, consider the quadratic equation $y^2 = 72$. When we take the square root — and when we remember that 72 has <u>two</u> square roots — we see that

$$y = \pm\sqrt{72} = \pm\sqrt{36 \cdot 2} = \pm\sqrt{36} \cdot \sqrt{2} = \pm\boldsymbol{6\sqrt{2}}$$

Now for the general statement: Assuming $A \geq 0$,

The solutions of the equation $x^2 = A$ are $x = \pm\sqrt{A}$

The Square Root Theorem

Notes:

1) The value of A in the Square Root Theorem is assumed to be zero or positive; otherwise, the square root of A will not be a real number, and therefore will be of no use to us in this course.

2) How do students usually mess up this kind of equation? By forgetting to include <u>both</u> square roots (that is, they forget the " $\pm$ " sign, ☹).

<u>EXAMPLE 1:</u> **Solve the quadratic equation: $(x + 7)^2 = 81$**

<u>Solution:</u> According to the Square Root Theorem, we can remove the squaring by taking the square <u>root</u> of both sides of the equation, remembering that the number 81 has <u>two</u> square roots:

$(x+7)^2 = 81$ (the original equation)

$\Rightarrow \quad x+7 = \pm\sqrt{81}$ (the Square Root Theorem)

$\Rightarrow \quad x+7 = \pm 9$ ($\sqrt{81} = 9$)

$\Rightarrow \quad x = -7 \pm 9$ (subtract 7 from each side)

Using the plus sign yields $x = -7 + 9 = 2$.

Using the minus sign yields $x = -7 - 9 = -16$.

$$\boxed{x = 2 \text{ or } -16}$$

<u>EXAMPLE 2:</u> **Solve for y: $(y - 3)^2 = 32$**

<u>Solution:</u> First we need to remove, or undo, the squaring in this quadratic equation. This is where we apply the Square Root Theorem, noting that 32 has two square roots:

$$y - 3 = \pm\sqrt{32}$$

To isolate the y, add 3 to both sides:

$$y = 3 \pm \sqrt{32}$$

Since $\sqrt{32} = \sqrt{16 \times 2} = \sqrt{16} \times \sqrt{2} = 4\sqrt{2}$, we simplify our solution:

$$\boxed{y = 3 \pm 4\sqrt{2}}$$

Be sure it's clear that we have written <u>two</u> solutions for our quadratic equation: $3 + 4\sqrt{2}$ and $3 - 4\sqrt{2}$.

<u>EXAMPLE 3:</u> **Solve for z:** $\left(z - \frac{4}{5}\right)^2 = \frac{8}{25}$

<u>Solution:</u> Using the Square Root Theorem, we take the square root of each side of the equation, remembering that $\frac{8}{25}$ has <u>two</u> square roots:

$$z - \frac{4}{5} = \pm\sqrt{\frac{8}{25}} \qquad \text{(the Square Root Theorem)}$$

$$\Rightarrow \quad z = \frac{4}{5} \pm \sqrt{\frac{8}{25}} \qquad \text{(isolate the } z\text{)}$$

$$\Rightarrow \quad z = \frac{4}{5} \pm \frac{\sqrt{8}}{\sqrt{25}} \qquad \text{(split the radical)}$$

$$\Rightarrow \quad z = \frac{4}{5} \pm \frac{2\sqrt{2}}{5} \qquad \text{(simplify both square roots)}$$

$$\Rightarrow \quad \boxed{z = \frac{4 \pm 2\sqrt{2}}{5}} \qquad \text{(combine into a single fraction)}$$

EXAMPLE 4: **Solve for n: $(n-3)^2 = -49$**

Solution: Applying the Square Root Theorem to remove the squaring gives us the equation

$$n-3 = \pm\sqrt{-49}$$

We needn't go any further; after all, the square root of a negative number doesn't exist in this class (it's not a real number). So we're done right here, and we conclude that the equation has

No Solution

Homework

10. Solve each equation by applying the Square Root Theorem:

a. $x^2 = 144$ b. $y^2 = 51$ c. $z^2 = 72$

d. $a^2 = 0$ e. $b^2 = -9$ f. $(x+1)^2 = 25$

g. $(n-3)^2 = 100$ h. $(u+10)^2 = 1$ i. $(a-5)^2 = 32$

j. $(b+7)^2 = 50$ k. $(w+13)^2 = -4$ l. $(m-3)^2 = 75$

11. Solve each equation by applying the Square Root Theorem:

a. $\left(x-\frac{1}{2}\right)^2 = \frac{3}{4}$ b. $\left(t+\frac{2}{3}\right)^2 = \frac{1}{9}$ c. $\left(z-\frac{4}{5}\right)^2 = \frac{19}{25}$

e. $\left(x+\frac{3}{5}\right)^2 = \frac{12}{25}$ f. $\left(b-\frac{9}{10}\right)^2 = \frac{81}{100}$ g. $\left(g-\frac{3}{7}\right)^2 = \frac{24}{49}$

Solutions

1. a. 3, 9, 17 b. 2, –8, 14 c. 6, 2, –2 d. 5, –13, –1
e. 12, 0, 13 f. 1, –13, 0 g. 1, –1, 1 h. –2, 0, –19
i. –1, 14, 0 j. –1, 0, –99 k. 2, –4, –3 l. –3, –2, 5
m. 1, –7, 4 n. 6, –1, –1 o. 18, 0, 0 p. –3, 3, 0

2. a. 145 b. 8 c. 1 d. 25 e. 0 f. –4

3. a. 7 b. –3 c. 7 d. Does not exist e. 8
f. 1 g. 0 h. 6 i. Does not exist j. 9

4. a. –13 b. 5 c. –7 d. –7 e. 1 f. 0

5. a. 121, –121 b. 11, –11 c. 0 d. Does not exist

6. a. 107 b. –16 c. $\frac{7}{2}$ d. –1 e. 0

f. 0 g. $\frac{4}{3}, 0$ h. $\frac{8}{9}, -\frac{10}{9}$

7. a. $b^2 - 4ac$

$= (\ )^2 - 4(\ )(\ )$

Converting each variable to a set of parentheses is a handy way to make sure everything is written properly.

$= (-5)^2 - 4(-10)(3)$

Note: The parentheses around the –5 are required, because as we've learned, $(-5)^2 = 25$, while the expression -5^2 is equal to –25.

$= 25 - (-120) = 25 + 120 = 145$

b. 52 c. –140 d. 9

8. a.

$$\pm\sqrt{b^2-4ac}$$
$$= \pm\sqrt{(\ \ )^2-4(\ \ )(\ \ )}$$
$$= \pm\sqrt{(-7)^2-4(2)(-15)}$$
$$= \pm\sqrt{49-(-120)}$$
$$= \pm\sqrt{49+120}$$
$$= \pm\sqrt{169}$$
$$= \pm 13$$

b. 0 c. ± 18 d. ± 3

9. a. $(x+5)^2$ b. $(y-9)^2$ c. $\left(a+\frac{1}{2}\right)^2$ d. $\left(m-\frac{2}{3}\right)^2$

e. $\left(z+\frac{1}{5}\right)^2$ f. $\left(w-\frac{5}{6}\right)^2$ g. $\left(b+\frac{9}{10}\right)^2$ h. $\left(u+\frac{3}{4}\right)^2$

i. $\left(n-\frac{2}{7}\right)^2$ j. $\left(x+\frac{5}{11}\right)^2$

10. a. $x = \pm 12$ b. $y = \pm\sqrt{51}$ c. $z = \pm 6\sqrt{2}$

d. $a = 0$ e. No solution in $\mathbb{R}$ f. $x = 4, -6$

g. $n = 13, -7$ h. $u = -9, -11$ i. $a = 5 \pm 4\sqrt{2}$

j. $b = -7 \pm 5\sqrt{2}$ k. No solution in $\mathbb{R}$ l. $m = 3 \pm 5\sqrt{3}$

11. a. $\left(x-\frac{1}{2}\right)^2 = \frac{3}{4} \Rightarrow x-\frac{1}{2} = \pm\sqrt{\frac{3}{4}} \Rightarrow x = \frac{1}{2} \pm \frac{\sqrt{3}}{\sqrt{4}} = \frac{1}{2} \pm \frac{\sqrt{3}}{2} = \frac{1 \pm \sqrt{3}}{2}$

b. $t = -\frac{1}{3}, -1$ c. $z = \frac{4 \pm \sqrt{19}}{5}$ d. No solution in $\mathbb{R}$

e. $x = \frac{-3 \pm 2\sqrt{3}}{5}$ f. $b = \frac{9}{5}, 0$ g. $g = \frac{3 \pm 2\sqrt{6}}{7}$

h. $p = \frac{-4 \pm 3\sqrt{7}}{9}$

"*Education* is what you get when you read the fine print. *Experience* is what you get if you don't."

– ***Pete Seeger***